Variational principle for frozen-in vorticity interacting with sound waves

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General properties of conservative hydrodynamic-type models are treated from positions of the canonical formalism adopted for liquid continuous media. A variational formulation is found for motion and interaction of frozen-in localized vortex structures and acoustic waves in a special description where dynamical variables are, besides the Eulerian fields of the fluid density and the potential component of the canonical momentum, also the shapes of frozen-in lines of the generalized vorticity. This variational principle can serve as a basis for approximate dynamical models with reduced number of degrees of freedom.

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In many physical systems the motion of continuous liquid media can be approximately described by hydrodynamic-type equations having a remarkable mathematical structure based on an underlying variational least action principle in the Lagrangian description (see, for instance, Refs. [1–6], and references therein). The characteristic feature of the hydrodynamic-type systems is that they possess an infinite number of specific integrals of motion related to the freezing-in property of canonical vorticity. Thus, hydrodynamic equations describe an interaction between "soft" degrees of freedom, the frozen-in vortices, and "hard" degrees of freedom, the acoustic modes. However, in the Eulerian description of flows by the density and by the velocity (or the canonical momentum) fields, the vorticity and the sound waves are "mixed." Another point is that due to unresolved freezing-in constraints, the Eulerian equations of motion do not follow directly from a variational principle (see Ref. [4] for discussion).

This work has two main purposes. The first purpose is to introduce such a general description of ideal flows that soft and hard degrees of freedom are explicitly separated and the frozen-in property of the vorticity is taken into account. The second purpose is to formulate a principle of least action in this representation. As a result, the acoustic waves will be described by the Eulerian fields of the fluid density and the potential component of the canonical momentum, while the canonical vorticity will be represented as a continuous distribution of frozen-in vortex lines (the so-called formalism of vortex lines [7–10], which previously was applied only to static density profiles). The Lagrangian of this dynamical system is a nontrivial unification of the canonical Lagrangian corresponding to purely potential flows, with a generalized Lagrangian of vortex lines.

Generalized Euler equation. Typically in a complex classical system on the microscopic level there are permanently existing particles of several kinds, for instance, molecules in a gas or the electrons and ions in a plasma. In a general situation, different components can have different macroscopically averaged velocities near the same point and/or different relative concentrations in separated points. In such cases, each population of the complex fluid should be taken into consideration individually, for example, as in the widely known two-fluid plasma model discussed later in this work. For simplicity we first consider the case when the macroscopic velocities of all components coincide and mutual relations between the concentrations are homogeneous in space and time, so the macroscopically averaged physical state of the medium at a given point $\boldsymbol{r}=(x,y,z)$ at a given time moment t is completely determined by two quantities, namely, by a scalar $n(\boldsymbol{r},t)$, which is proportional to the concentration of conservative particles of a definite sort, and by a vector $\boldsymbol{j}(\boldsymbol{r},t)$, the corresponding density of flow. The field $\boldsymbol{j}$ is related by the continuity equation to the field n:

$$n_t+\operatorname{div}\boldsymbol{j}=0, \tag{1}$$

where the subscript is used to denote the partial derivative. It is clear that $\boldsymbol{j}=n\boldsymbol{v}$, where $\boldsymbol{v}(\boldsymbol{r},t)$ is the macroscopic velocity field. Let each point of the fluid medium be marked by a label $\boldsymbol{a}=(a_1,a_2,a_3)$, so the mapping $\boldsymbol{r}=\boldsymbol{x}(\boldsymbol{a},t)$ is the full Lagrangian description of the flow. The less exhaustive description of the flow by the fields $n(\boldsymbol{r},t)$ and $\boldsymbol{j}(\boldsymbol{r},t)$ is commonly referred as the Eulerian description. The relations between the Eulerian fields and the Lagrangian mapping are the following:

$$n(\boldsymbol{r},t)=\int \delta(\boldsymbol{r}-\boldsymbol{x}(\boldsymbol{a},t))d\boldsymbol{a}, \tag{2}$$

$$\boldsymbol{j}(\boldsymbol{r},t)=\int \delta(\boldsymbol{r}-\boldsymbol{x}(\boldsymbol{a},t))\boldsymbol{x}_t(\boldsymbol{a},t)d\boldsymbol{a}, \tag{3}$$

and they satisfy the continuity equation automatically.

Neglecting all dissipative processes (due to viscosity, diffusion, etc.) and assuming that internal properties of the fluid are homogeneous (such as the specific entropy in adiabatic flows or the temperature in isothermal flows), the trajectories $\boldsymbol{r}=\boldsymbol{x}(\boldsymbol{a},t)$ of fluid elements are determined by the variational principle $\delta(\int\tilde{\mathcal{L}}dt)/\delta\boldsymbol{x}(\boldsymbol{a},t)=0$, with the Lagrangian of a special general form actually depending only on the Eulerian fields $n(\boldsymbol{r},t)$ and $\boldsymbol{j}(\boldsymbol{r},t)$:

$$\tilde{\mathcal{L}}\{\boldsymbol{x}(\boldsymbol{a}),\boldsymbol{x}_t(\boldsymbol{a})\}=\mathcal{L}\{n(\boldsymbol{r}),\boldsymbol{j}(\boldsymbol{r})\}\big|_{n\{\mathbf{x}\},j\{\mathbf{x},\mathbf{x}_t\}}. \tag{4}$$

Here the braces $\{\cdots\}$ are used to denote functional arguments as against usual scalar or vector arguments that are denoted by the parentheses $(\cdots)$. The equation of motion, corresponding to Lagrangian (4), has a remarkable general structure. The usual variational Euler-Lagrange equation

$$\frac{d}{dt}\frac{\delta\tilde{\mathcal{L}}}{\delta \boldsymbol{x}_t(\boldsymbol{a})}=\frac{\delta\tilde{\mathcal{L}}}{\delta \boldsymbol{x}(\boldsymbol{a})}$$

in the Eulerian representation has the form (generalized Euler equation)

$$\frac{\partial}{\partial t}\left(\frac{\delta\mathcal{L}}{\delta\boldsymbol{j}}\right)=\left[\frac{\boldsymbol{j}}{n}\times\mathrm{curl}\left(\frac{\delta\mathcal{L}}{\delta\boldsymbol{j}}\right)\right]+\boldsymbol{\nabla}\left(\frac{\delta\mathcal{L}}{\delta n}\right), \tag{5}$$

where the variational derivative $\delta\mathcal{L}/\delta\boldsymbol{j}$ is taken at fixed $n(\boldsymbol{r},t)$, while the variational derivative $\delta\mathcal{L}/\delta n$ is taken at fixed $\boldsymbol{j}(\boldsymbol{r},t)$ (compare with Refs. [6–9], where this equation is written in terms of n and $\boldsymbol{v}$ and thus looks different). Equation (5) together with the continuity equation (1) completely determine the time evolution of the fields $n(\boldsymbol{r},t)$ and $\boldsymbol{j}(\boldsymbol{r},t)$.

Hamiltonian variables. In the Hamiltonian description adopted for fluids as it is discussed in Refs. [7–9,11], instead of the field $\boldsymbol{j}$, the variational derivative of the Lagrangian, $\boldsymbol{p}=\delta\mathcal{L}/\delta\boldsymbol{j}$, is used (the canonical momentum). The Hamiltonian functional is defined as the Legendre transformation

$$\mathcal{H}\{n,\boldsymbol{p}\}\equiv\int\left(\frac{\delta\mathcal{L}}{\delta\boldsymbol{j}}\cdot\boldsymbol{j}\right)d\boldsymbol{r}-\mathcal{L}, \tag{6}$$

where $\boldsymbol{j}$ should be expressed in terms of $\boldsymbol{p}$ and n. The equations of motion (5) and (1) now have the noncanonical Hamiltonian structure [4,7,8]:

$$\boldsymbol{p}_t=\left[\frac{1}{n}\left(\frac{\delta\mathcal{H}}{\delta\boldsymbol{p}}\right)\times\mathrm{curl}\,\boldsymbol{p}\right]-\boldsymbol{\nabla}\left(\frac{\delta\mathcal{H}}{\delta n}\right), \tag{7}$$

$$n_t=-\mathrm{div}\left(\frac{\delta\mathcal{H}}{\delta\boldsymbol{p}}\right). \tag{8}$$

These equations can be written as $\boldsymbol{p}_t=\{\boldsymbol{p},\mathcal{H}\}$ and $n_t=\{n,\mathcal{H}\}$, where the noncanonical Poisson bracket is given by the following expression (see Refs. [2,4,7] and references therein about details):

$$\{\mathcal{F},\mathcal{H}\}=\int\left[\frac{\delta\mathcal{H}}{\delta n}\left(\boldsymbol{\nabla}\cdot\frac{\delta\mathcal{F}}{\delta\boldsymbol{p}}\right)-\frac{\delta\mathcal{F}}{\delta n}\left(\boldsymbol{\nabla}\cdot\frac{\delta\mathcal{H}}{\delta\boldsymbol{p}}\right)\right]d\boldsymbol{r}+\int\left(\frac{\mathrm{curl}\,\boldsymbol{p}}{n}\cdot\left[\frac{\delta\mathcal{F}}{\delta\boldsymbol{p}}\times\frac{\delta\mathcal{H}}{\delta\boldsymbol{p}}\right]\right)d\boldsymbol{r}. \tag{9}$$

First example: Eulerian hydrodynamics. Let us consider the usual Eulerian hydrodynamics. In this simple case, n is the density of the fluid and the Lagrangian is the difference between the total macroscopic kinetic energy and the total potential energy including the thermal internal energy:

$$\mathcal{L}_E=\int\left(\frac{\boldsymbol{j}^2}{2n}-\varepsilon(n)-nU(\boldsymbol{r},t)\right)d\boldsymbol{r},$$

where $\varepsilon(n)$ is the density of the internal energy and $U(\boldsymbol{r},t)$ is the potential of an external force.

The canonical momentum coincides with the velocity field, $\boldsymbol{p}=\boldsymbol{j}/n=\boldsymbol{v}$, and the Hamiltonian is the total energy expressed in terms of n and $\boldsymbol{p}$:

$$\mathcal{H}_E=\int\left(n\frac{\boldsymbol{p}^2}{2}+\varepsilon(n)+nU(\boldsymbol{r},t)\right)d\boldsymbol{r}.$$

The equations of motion (7) and (8) with this Hamiltonian take the known form

$$\boldsymbol{p}_t=[\boldsymbol{p}\times\mathrm{curl}\,\boldsymbol{p}]-\boldsymbol{\nabla}(\boldsymbol{p}^2/2+\varepsilon'(n)+U),\quad n_t+\mathrm{div}(n\boldsymbol{p})=0.$$

Second example: relativistic fluid dynamics. In the general relativity the continuity equation is (see Ref. [1])

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}\tilde{n}\frac{dx^i}{ds}\right)=0,$$

where dx^i/ds is the 4-velocity of the fluid element passing through a point $(t,\boldsymbol{r})$, the scalar $\tilde{n}$ is the concentration of conservative particles in the locally comoving frame of reference, and $g=\det\|g_{ik}\|$ is the determinant of the metric tensor $g_{ik}(t,\boldsymbol{r})$. Therefore $n=\sqrt{-g}\tilde{n}(dt/ds)$, $j^\alpha=\sqrt{-g}\tilde{n}(dx^\alpha/ds)$, and

$$\tilde{n}=\sqrt{(g_{00}n^2+2g_{0\alpha}nj^\alpha+g_{\alpha\beta}j^\alpha j^\beta)}/\sqrt{-g}. \tag{10}$$

The invariant expression for the action functional implies the Lagrangian in the form (compare with Refs. [7,9,10])

$$\mathcal{L}_{\mathrm{g.r.}}=-\int\varepsilon\left(\frac{\sqrt{g_{00}n^2+2g_{0\alpha}nj^\alpha+g_{\alpha\beta}j^\alpha j^\beta}}{\sqrt{-g}}\right)\sqrt{-g}\,d\boldsymbol{r}, \tag{11}$$

where $\varepsilon(\tilde{n})$ is the relativistic density of the internal fluid energy including the rest energy (the equation of state). The canonical momentum

$$p_\alpha=\varepsilon'(\tilde{n})\frac{-(g_{0\alpha}n+g_{\alpha\beta}j^\beta)}{\sqrt{g_{00}n^2+2g_{0\alpha}nj^\alpha+g_{\alpha\beta}j^\alpha j^\beta}}$$

depends in a complicated manner on n, $\boldsymbol{j}$, and g_{ik}. This makes it impossible, in general case, to present an analytical expression for the corresponding Hamiltonian functional, but, of course, it cannot cancel the existence of the Hamiltonian in mathematical sense.

Third example: Lagrangian functional of the two-fluid plasma model. Analogously, multicomponent hydrodynamical models can be investigated, where several fields $n^{(q)}$ and $\boldsymbol{j}^{(q)}$ are present corresponding to different sorts of particles, with $q=1,2,\ldots,Q$. The Hamiltonian noncanonical equations of motion for such models have the same general structure as discussed above, and they should be written for each component. Here we briefly discuss how to derive the Lagrangian functional for the physically important two-fluid (nonrelativistic) plasma model (for more details see Ref. [11]). Let us consider a set of electrically charged classical point particles with masses m_a and electric charges e_a. In Ref. [1], the microscopic Lagrangian of such a system is presented, which is approximately valid up to the second order in v/c, since excitation of the free electromagnetic field by moving charges is not significant:

$$\mathcal{L}_{\rm micro}=\sum_a \frac{m_a\boldsymbol{v}_a^2}{2}-\frac{1}{2}\sum_{a\neq b}\frac{e_ae_b}{|\boldsymbol{r}_a-\boldsymbol{r}_b|}+\sum_a \frac{m_a\boldsymbol{v}_a^4}{8c^2}+\frac{1}{4c^2}\sum_{a\neq b}\frac{e_ae_b}{|\boldsymbol{r}_a-\boldsymbol{r}_b|}[\boldsymbol{v}_a\cdot\boldsymbol{v}_b+(\boldsymbol{v}_a\cdot\boldsymbol{n}_{ab})(\boldsymbol{v}_b\cdot\boldsymbol{n}_{ab})]. \tag{12}$$

Here $\boldsymbol{r}_a(t)$ are the positions of the point charges e_a, $\boldsymbol{v}_a(t)\equiv\dot{\boldsymbol{r}}_a(t)$ are their velocities, and $\boldsymbol{n}_{ab}(t)$ are the unit vectors in the direction between e_a and e_b. The first double sum in Eq. (12) corresponds to the electrostatic interaction, while the second double sum describes the magnetic interaction via quasistationary magnetic field (the case without external electric and magnetic fields is considered). It is important that for a system with huge number of particles the magnetic energy can be of the same order (or even larger) as the macroscopic kinetic energy produced by the first ordinary sum in Eq. (12), while the terms of the fourth order on the velocities are often negligible.

In the simplest case, plasma contains identical ions (each with the mass ZM and with the charge $+Ze$) and the electrons (mass m, charge $-e$). Then, the system will be approximately described in terms of the concentration $n(\boldsymbol{r},t)$ of electrons and the density $\boldsymbol{j}(\boldsymbol{r},t)$ of their flow, and corresponding ion fields $N(\boldsymbol{r},t)$ and $\boldsymbol{J}(\boldsymbol{r},t)$, normalized to the elementary electric charge e, so $N=ZN_i$.

Neglecting all dissipative processes that take place due to collisions of the particles, we derive from Eq. (12) the following Lagrangian functional

$$\mathcal{L}_{\rm 2f}=\int\left[\frac{M}{2N}\boldsymbol{J}^2+\frac{m}{2n}\boldsymbol{j}^2+\frac{2\pi e^2}{c^2}[\mathrm{curl}^{-1}(\boldsymbol{J}-\boldsymbol{j})_\perp]^2\right]d\boldsymbol{r}-\frac{e^2}{2}\int\int\frac{d\boldsymbol{r}_1d\boldsymbol{r}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}[N(\boldsymbol{r}_1)-n(\boldsymbol{r}_1)][N(\boldsymbol{r}_2)-n(\boldsymbol{r}_2)]-\int\left[T_e n\ln\frac{n}{f(T_e)}+T_i\frac{N}{Z}\ln\frac{N}{ZF(T_i)}\right]d\boldsymbol{r}, \tag{13}$$

where $(\boldsymbol{J}-\boldsymbol{j})_\perp$ is the divergence-free component of the total current. The magnetic energy $\int(\boldsymbol{B}^2/8\pi)d\boldsymbol{r}$ is included into this Lagrangian, where the magnetic field is $\boldsymbol{B}=(4\pi e/c)\mathrm{curl}^{-1}(\boldsymbol{J}-\boldsymbol{j})_\perp$. The terms with $T_e n\ln n$ and $(T_i/Z)N\ln N$ (approximate expressions for the densities of the thermal free energy) have been introduced in order that the macroscopic equations of motion contain the pressure terms such as $-\nabla p/n$ [see the last term in Eq. (5)], where $p\approx nT_e$ is the pressure of the hot electron gas, which is supposed to be isothermal with a temperature T_e.

Interaction between frozen-in vortex lines and acoustic modes. The Hamiltonian noncanonical equations (7) and (8) do not follow directly from a variational principle. The mathematical reason for this is a degeneracy of the corresponding noncanonical Poisson bracket (9), which is discussed, for instance, in Refs. [2,4]. The degeneracy results in the frozen-in property for the canonical vorticity field $\boldsymbol{\omega}=\mathrm{curl}\,\boldsymbol{p}$. However, representations of the canonical momentum in terms of auxiliary variables exist that fix topological structure of vortex lines, and then a variational formulation becomes possible. A known example of such auxiliary variables is the Clebsch variables (see, e.g., Refs. [2,4,12,13] for discussion and more references), when $\boldsymbol{p}=\nabla\varphi+(\lambda/n)\nabla\mu$, and (n,φ), (λ,μ) are two pairs of canonically conjugate variables. But the Clebsch representation usually is not suitable for studying localized vortex structures such as vortex filaments. Below we consider another representation for the canonical momentum field, when dynamical variables are the shapes of vortex lines. For a nearly static density profile, $n\approx n_0(\boldsymbol{r})$, such description was used in Refs. [8,9] to study slow flows in spatially inhomogeneous systems. Now we are going to introduce a variational formulation valid for the general case, since the function $n(\boldsymbol{r},t)$ is also an unknown variable. It will be demonstrated that variational principle with Lagrangian (23) determines correct equations of motion for the shapes of frozen-in vortex lines, for the potential component of the canonical momentum field, and for the density profile $n(\boldsymbol{r},t)$.

So, we decompose the momentum field onto the potential component and the divergence-free component:

$$\boldsymbol{p}(\boldsymbol{r},t)=\nabla\varphi(\boldsymbol{r},t)+\mathrm{curl}^{-1}\boldsymbol{\omega}(\boldsymbol{r},t). \tag{14}$$

Accordingly, the field $\boldsymbol{j}$ is decomposed:

$$\boldsymbol{j}=\frac{\delta\mathcal{H}}{\delta\boldsymbol{p}}=\frac{\delta\mathcal{H}}{\delta\boldsymbol{p}_\parallel}+\mathrm{curl}\frac{\delta\mathcal{H}}{\delta\boldsymbol{\omega}}\equiv\boldsymbol{j}_\parallel+\boldsymbol{j}_\perp. \tag{15}$$

Obviously, the continuity equation results in the relation

$$\nabla\Delta^{-1}n_t=-\boldsymbol{j}_\parallel. \tag{16}$$

For the frozen-in vorticity field we use the so-called vortex line representation. In the simplest form when the lines are closed, it reads as follows (for details and discussion see Ref. [8]),

$$\boldsymbol{\omega}(\boldsymbol{r},t)=\int_{\mathcal{N}}d^2\nu\oint\delta(\boldsymbol{r}-\boldsymbol{R}(\nu,\xi,t))\boldsymbol{R}_\xi(\nu,\xi,t)\,d\xi=\left.\frac{\boldsymbol{R}_\xi(\nu,\xi,t)}{\det\|\partial\boldsymbol{R}/\partial(\nu,\xi)\|}\right|_{\mathbf{R}=\mathbf{r}}, \tag{17}$$

where the label $\nu=(\nu_1,\nu_2)\in\mathcal{N}$ belongs to a two-dimensional manifold $\mathcal{N}$ and singles out an individual vortex line, while an arbitrary longitudinal parameter ξ determines a point on the line. The Jacobian of the mapping $\boldsymbol{R}(\nu,\xi,t)$ is denoted as $\det\|\partial\boldsymbol{R}/\partial(\nu,\xi)\|=([\boldsymbol{R}_{\nu_1}\times\boldsymbol{R}_{\nu_2}]\cdot\boldsymbol{R}_\xi)$.

The divergence-free component of the canonical momentum field is now given by the expression

$$\boldsymbol{p}_\perp=\mathrm{curl}^{-1}\boldsymbol{\omega}(\boldsymbol{r},t)=\int\frac{[\boldsymbol{R}_\xi\times(\boldsymbol{r}-\boldsymbol{R})]d^2\nu d\xi}{4\pi|\boldsymbol{r}-\boldsymbol{R}|^3}. \tag{18}$$

The vorticity variation $\delta\boldsymbol{\omega}(\boldsymbol{r},t)$, induced by a variation $\delta\boldsymbol{R}(\nu,\xi,t)$ of the vortex lines, takes the form [8]

$$\delta\boldsymbol{\omega}(\mathbf{r},t)=\mathrm{curl}_{\mathbf{r}}\int_{\mathcal{N}}d^2\nu\oint\delta(\boldsymbol{r}-\boldsymbol{R}(\nu,\xi,t))[\delta\boldsymbol{R}\times\boldsymbol{R}_\xi]\,d\xi,\quad(19)$$

which follows directly from Eq. (17).

It should be noted that in the case of arbitrary topology of the vortex lines, one has to just replace in the above expressions $\boldsymbol{R}(\nu,\xi,t)\rightarrow\boldsymbol{R}(\boldsymbol{a},t)$ and $\boldsymbol{R}_\xi d^2\nu\, d\xi \rightarrow(\boldsymbol{\omega}_0(\boldsymbol{a})\cdot\boldsymbol{\nabla}_a)\boldsymbol{R}(\boldsymbol{a},t)d\boldsymbol{a}$, see Ref. [8].

Equation (19) results in the important relations [8]

$$\frac{\delta\mathcal{H}}{\delta\boldsymbol{R}}=[\boldsymbol{R}_\xi\times\boldsymbol{j}_\perp(\boldsymbol{R})],\quad(20)$$

$$\boldsymbol{\omega}_t=\mathrm{curl}_{\mathbf{r}}\left[\frac{\boldsymbol{R}_t\times\boldsymbol{R}_\xi}{\det\|\partial\boldsymbol{R}/\partial(\nu,\xi)\|}\right]\Bigg|_{\mathbf{R}=\mathbf{r}}.\quad(21)$$

Therefore the equation of motion for the vorticity, $\boldsymbol{\omega}_t=\mathrm{curl}_{\mathbf{r}}[\boldsymbol{v}\times\boldsymbol{\omega}]$, with $\boldsymbol{v}=(\delta\mathcal{H}/\delta\boldsymbol{p})/n$, means

$$\left[\frac{\boldsymbol{R}_t\times\boldsymbol{R}_\xi}{\det\|\partial\boldsymbol{R}/\partial(\nu,\xi)\|}\right]\Bigg|_{\mathbf{R}=\mathbf{r}}=\left[\frac{\boldsymbol{j}_\parallel+\boldsymbol{j}_\perp}{n}\times\boldsymbol{\omega}\right]+\boldsymbol{\nabla}_{\mathbf{r}}\Psi(\nu),\quad(22)$$

where $\Psi(\nu_1,\nu_2)$ is some arbitrary function of two variables. A possible choice is $\Psi=0$, but for general purposes we will consider below $\Psi\neq0$.

Using Eqs. (15), (16), (20), and (22), one can verify that if the quantities $\boldsymbol{R}(\nu,\xi,t)$, $n(\boldsymbol{r},t)$, and $\varphi(\boldsymbol{r},t)$ obey equations of motion corresponding to the following Lagrangian:

$$\mathcal{L}_{\text{v-s}}=-\int n\varphi_t d\boldsymbol{r}-\mathcal{H}\{n,\boldsymbol{\nabla}\varphi+\mathrm{curl}^{-1}\boldsymbol{\omega}\{\boldsymbol{R}\}\}$$
$$+\int\delta(\boldsymbol{r}-\boldsymbol{R}(\nu,\xi,t))([\boldsymbol{R}_\xi\times\boldsymbol{R}_t]\cdot\boldsymbol{\nabla}_{\mathbf{r}}\Delta_{\mathbf{r}}^{-1}n)d^2\nu\,d\xi\,d\boldsymbol{r}$$
$$-\int\Psi(\nu_1,\nu_2)([\boldsymbol{R}_{\nu_1}\times\boldsymbol{R}_{\nu_2}]\cdot\boldsymbol{R}_\xi)n(\boldsymbol{R})d^2\nu\,d\xi,\quad(23)$$

then Eqs. (7) and (8) are satisfied. Indeed, the variation of $\int\mathcal{L}_{\text{v-s}}dt$ by $\delta\boldsymbol{R}(\nu,\xi,t)$ gives the equation

$$[\boldsymbol{R}_\xi\times\boldsymbol{R}_t]\,n(\boldsymbol{R})-[\boldsymbol{R}_\xi\times\boldsymbol{j}_\parallel(\boldsymbol{R})]$$
$$=\frac{\delta\mathcal{H}}{\delta\boldsymbol{R}}-n(\boldsymbol{R})\det\left\|\frac{\partial\boldsymbol{R}}{\partial(\nu,\xi)}\right\|\nabla_{\mathbf{r}}\Psi(\nu),\quad(24)$$

which is easily recognized as Eq. (22). Variation by $\delta n(\boldsymbol{r},t)$ results in the potential component of Eq. (7):

$$-\varphi_t+\Delta_{\mathbf{r}}^{-1}\boldsymbol{\nabla}_{\mathbf{r}}\cdot\left[\frac{\boldsymbol{R}_t\times\boldsymbol{R}_\xi}{\det\|\partial\boldsymbol{R}/\partial(\nu,\xi)\|}\right]\Bigg|_{\mathbf{R}=\mathbf{r}}=\frac{\delta\mathcal{H}}{\delta n}+\Psi.$$

Finally, the variation by $\delta\varphi(\boldsymbol{r},t)$ gives the continuity equation $n_t+\boldsymbol{\nabla}\cdot\boldsymbol{j}_\parallel=0$.

Discussion. Thus, Lagrangian (23) gives a required variational formulation for the problem of motion and interaction between localized frozen-in vortex structures [described by the mapping $\boldsymbol{R}(\nu,\xi,t)$] and acoustic degrees of freedom [described by the fields $n(\boldsymbol{r},t)$ and $\varphi(\boldsymbol{r},t)$]. This variational principle definitely can serve as a basis for future approximate analytical and numerical studies dealing with reduced dynamical systems where only most relevant degrees of freedom will be taken into account.

The function $\Psi(\nu_1,\nu_2)$ can be useful to investigate nearly stationary flows, since the effective Hamiltonian

$$\tilde{\mathcal{H}}=\mathcal{H}+\int\Psi(\nu_1,\nu_2)([\boldsymbol{R}_{\nu_1}\times\boldsymbol{R}_{\nu_2}]\cdot\boldsymbol{R}_\xi)n(\boldsymbol{R})d^2\nu\,d\xi$$

has an extremum on stationary flows with the velocity field $\boldsymbol{v}$ everywhere directed along vortex surfaces. However, one should remember that existence of globally defined vortex surfaces (and thus the function Ψ) is an exceptional case in the variety of three-dimensional vector fields. In the general case one should use $(\boldsymbol{\omega}_0(\boldsymbol{a})\cdot\boldsymbol{\nabla}_a)\boldsymbol{R}(\boldsymbol{a},t)d\boldsymbol{a}$ instead of $\boldsymbol{R}_\xi(\nu,\xi,t)d^2\nu d\xi$ in Lagrangian (23) and no function $\Psi(\nu_1,\nu_2)$, since the labels ν are not defined.

As an explicit example, the expression for the Hamiltonian functional of the Eulerian hydrodynamics in terms of $\boldsymbol{R}(\nu,\xi,t)$, $n(\boldsymbol{r},t)$, and $\varphi(\boldsymbol{r},t)$ is given below:

$$\mathcal{H}_E=\int\frac{n}{2}\left(\boldsymbol{\nabla}\varphi+\int\frac{[\boldsymbol{R}_\xi\times(\boldsymbol{r}-\boldsymbol{R})]d^2\nu d\xi}{4\pi|\boldsymbol{r}-\boldsymbol{R}|^3}\right)^2d\boldsymbol{r}$$
$$+\int[\varepsilon(n)+nU(\boldsymbol{r},t)]d\boldsymbol{r}.\quad(25)$$

Generalization of the above theory for multifluid models is straightforward.

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